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Existence, Uniqueness, Stability, and High-Order Numerical Approximation of Solutions to Nonlinear Fractional Differential Equations

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التفرد والثبات والاستقرار والتقريب العددي عالي الرتبة لحلول المعادلات التفاضلية الكسرية غير الخطية

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Abstract:
This study aims to formulate a unified framework for the analytical, numerical, and conceptual solution of nonlinear fractional differential equations of the Caputo type, which are characterized by weak initial regularity and the presence of fractional-power singularities near $t=0$. The study addresses issues concerning solvability, existence, uniqueness, and stability, while also investigating the phenomenon of a loss in convergence accuracy within standard high-order algorithms. The approach integrates Banach's fixed-point theorem, stability estimates based on weighted norms, and a convolution quadrature algorithm-specifically the second-order backward differentiation formula (BDF2)-augmented with corrections at the starting point.

Analytical results and mathematical bounds establish the existence, continuity, and uniqueness of solutions over finite time intervals of arbitrary length, under global Lipschitz assumptions; furthermore, the continuous dependence of solutions on initial values and residual perturbations is confirmed. It is demonstrated that terms involving fractional powers $(t-s)^{\alpha-1}$ are the primary cause of the reduction in the numerical order of the BDF2 scheme when applied to a standard problem with a fixed-power singularity. Consequently, corrections at the starting point are designed to restore second-order convergence accuracy, consistent with specified regularity requirements and expected consistency conditions. The corresponding unified error estimate demonstrates the feasibility of maintaining second-order accuracy, while accounting for temporal discretization errors, fractional-memory approximations, and nonlinear solution errors—with the latter two components controlled at the $\delta(\zeta)^{-\alpha} = \sum_{j\geq 0} \omega_j \zeta^j$ level. These findings address the conceptual, mathematical, and high-order numerical applicability aspects of Caputo-type fractional differential equations, while considering regularity properties. The study further recommends that these approaches be validated through testing against manufactured solutions and systematic comparative numerical studies.

Keywords : Nonlinear fractional differential equations; Caputo derivative; initial singularities; solution regularity; existence and uniqueness; stability; corrected convolution quadrature (BDF2); starting-point corrections; error estimation; convergence of numerical methods.

الملخص:
تهدف هذه الدراسة إلى صياغة إطار موحَّد للحل التحليلي والعددي والمفاهيمي للمعادلات التفاضلية الكسرية غير الخطية من نوع كابوتو(Caputo) ، التي تتميز بانتظامية ابتدائية ضعيفة ووجود شذوذات (تفرُّدات) ذات قوى كسرية بالقرب $t=0$.
وتتناول الدراسة قضايا قابلية الحل والوجود والوحدانية والاستقرار، كما تبحث ظاهرة فقدان دقة التقارب في الخوارزميات القياسية عالية الرتبة. ويجمع المنهج بين نظرية النقطة الثابتة لباناخ(Banach) ، وتقديرات الاستقرار المعتمدة على النُّظُم

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الموزونة، وخوارزمية التربيع الالتفافي(convolution quadrature) ، وتحديدًا صيغة الاشتقاق الخلفي من الرتبة الثانية (BDF2)، مدعومةً بتصحيحات عند نقطة البداية.
وتُثبت النتائج التحليلية والحدود الرياضية وجود الحلول واتصالها ووحدانيتها على فترات زمنية منتهية ذات طول اختياري، وذلك في ظل افتراضات ليبشيتز (Lipschitz) الشاملة؛ كما يتأكد الاعتماد المتصل للحلول على القيم الابتدائية وعلى الاضطرابات المتبقية. ويتضح أن الحدود التي تتضمن قوى كسرية من الشكل $(t-s)^{(\alpha-1)}$ هي السبب الرئيس في انخفاض الرتبة العددية لمخطط BDF2 عند تطبيقه على مسألة قياسية ذات تفرُّد بقوة ثابتة. ومن ثَمَّ صُمِّمت تصحيحات نقطة البداية لاستعادة دقة التقارب من الرتبة الثانية، بما يتوافق مع متطلبات الانتظامية المحددة وشروط الاتساق المتوقعة.
ويبيّن تقدير الخطأ الموحَّد المقابل إمكانية الحفاظ على الدقة من الرتبة الثانية، مع مراعاة أخطاء التقطيع الزمني، وتقريبات الذاكرة الكسرية، وأخطاء الحل غير الخطي، على أن يُضبط المكوّنان الأخيران عند مستوى $\delta(\zeta)^{(-\alpha)} = \Sigma_{(j\geq 0)} \omega_j \zeta^j$ وتعالج هذه النتائج الجوانب المفاهيمية والرياضية وإمكانية التطبيق العددي عالي الرتبة للمعادلات التفاضلية الكسرية من نوع كابوتو، مع الأخذ في الاعتبار خصائص الانتظامية. وتوصي الدراسة كذلك بالتحقق من صحة هذه المقاربات عبر الاختبار على حلول مُصنَّعة (manufactured solutions) وإجراء دراسات عددية مقارنة منهجية.

الكلمات المفتاحية: المعادلات التفاضلية الكسرية غير الخطية؛ مشتقة كابوتو؛ التفردات عند نقطة البداية؛ انتظام الحل؛ الوجود والوحدانية؛ الاستقرار؛ تكامل الالتفاف التربيعي المصحح (BDF2)؛ تصحيحات نقطة البداية؛ تقدير الخطأ؛ تقارب الطرق العددية.

1.Introduction

In recent years, fractional differential equations have attracted significant interest in mathematical fields because they can be used to accurately describe the behavior of complex dynamical systems exhibiting memory effects, hereditary properties, and nonlocal behavior in time. In particular, the fractional models can be used to describe processes whose evolution depends on the entire history rather than only on a contemporary state as traditional models[1]. This feature of fractional models makes them especially suitable for describing anomalous diffusion, viscoelastic materials, biological systems, engineering processes, and other systems with long memory[2]. The Caputo operator is most widely used among various types of fractional operators because the initial conditions can be described in terms of classical physical quantities. On the other hand, the solution of nonlinear Caputo fractional differential equations poses great challenges to the theories and methods because of the nonlinear source term, history-dependent operators, and weak solution regularity (possibly fractional power solution behavior as t^{α} near the initial point $t = 0$, despite the nonlinearity of source terms)[2,3,4]. The weak initial regularity, in particular, may significantly downgrade the convergence order of standard high-order numerical algorithms. Therefore, mathematically rigorous numerical algorithms that can incorporate initial singularities are desirable in the theory and application of fractional differential equations[5].

The main objective of this work is to formulate a comprehensive regularity-robust mathematical and numerical framework for the study of (1) the solvability, uniqueness, and stability of solutions to nonlinear Caputo fractional initial-value problems, and (2) the high-order numerical discretization for these solutions. In particular, the study aims to identify the sources of regularity deficiency of solutions and, consequently, the order reduction of high-order numerics in the temporal domain. The study further aims to establish conditions for the existence and uniqueness of solutions based on fixed-point theory, as well as their continuous dependence on initial conditions and perturbations. From a computational perspective, the study proposes a corrected backward differentiation formula-based convolution quadrature method (specifically, BDF2) with initial correction terms designed to mitigate weak regularity. Additional goals include providing a systematic approach to determine the optimal initial correction terms, developing temporal discretization error estimates that encapsulate the effects of temporal discretization, compressed fractional history, and the errors in a nonlinear algebraic

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solver, and subsequently providing a criterion for selecting tolerances such that the final order of convergence is retained. The significance of this research resides in the blend of rigorous theoretical analysis and computationally relevant numerical error control that will subsequently serve as a framework for the numerical solution of nonlinear Caputo fractional differential equations with limited temporal regularity.

Despite significant progress in the numerical approximation of fractional differential equations, a robust, generally applicable algorithmic design methodology enabling reliable high-order convergence of Caputo problems with weak initial regularity remains an open problem. While existing methods such as starting-weight corrections, graded temporal grids, and high-order convolution quadrature methods have highlighted the importance of handling initial singularities [8–10], the combined considerations of regularity-dependent correction selection, temporal approximation errors, fractional history compression, and nonlinear solver tolerances pose an open research question within an explicit formulation of the underlying numerical scheme. That question is whether convergence order can be preserved when applying a numerical scheme to a solution featuring fractional power singularities near the initial time, where additional operations introduced by computational approximations can induce errors that could potentially break the theoretical order of the scheme[7,9,11]. To answer this question, the project will develop a regularity-aware corrected BDF2 convolution quadrature scheme, with analytical descriptions of the singular terms, fixed-point existence arguments, and perturbation-based stability estimates; it will establish an integrated error-budget approach that balances the temporal discretization, the compressed-history approximation, and the nonlinear iterative solver tolerances; and it will design manufactured-solution tests and systematic convergence experiments to demonstrate the correction scheme in a reproducible manner. These are research objectives, not experimentally validated conclusions; novelty relative to existing corrected convolution-quadrature schemes requires a precise comparison and independent numerical verification..

2. Material nd Methods

The study employs a quantitative research methodology—mathematical, analytical, and computational—to develop a "Unified Regularity-Aware Framework" for solving nonlinear fractional differential equations of the Caputo type. It addresses a critical challenge in fractional numerical analysis: the loss of expected accuracy in high-order numerical methods, which stems from the limited regularity of the solution near the initial time ($t=0$). The core concept extends beyond merely developing a new numerical method; it integrates mathematical analysis of the solution, the identification of initial singularities, the correction of the numerical scheme, and the management of error sources into a cohesive, unified framework. This represents the novel contribution of the study.

2.1 Analytical framework

The analysis is built on the equivalent integral equation (2). Existence and uniqueness follow from the Banach fixed-point theorem in the weighted norm, which turns the memory term into a single contraction constant $\theta(\lambda) = L\,\lambda^{(-\alpha)}$ and yields global results without restrictions on T. Stability is studied in three senses: continuous dependence on the initial value, continuous dependence on perturbations of f, and residual stability (a function that satisfies (2) up to a small residual is close to the solution). The same three senses are then transferred to the discrete level, where the integral equation is replaced by a convolution-quadrature discretization whose

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integral weights require a separately justified positivity and asymptotic analysis, so that a discrete weighted-norm argument parallels the continuous one[12].

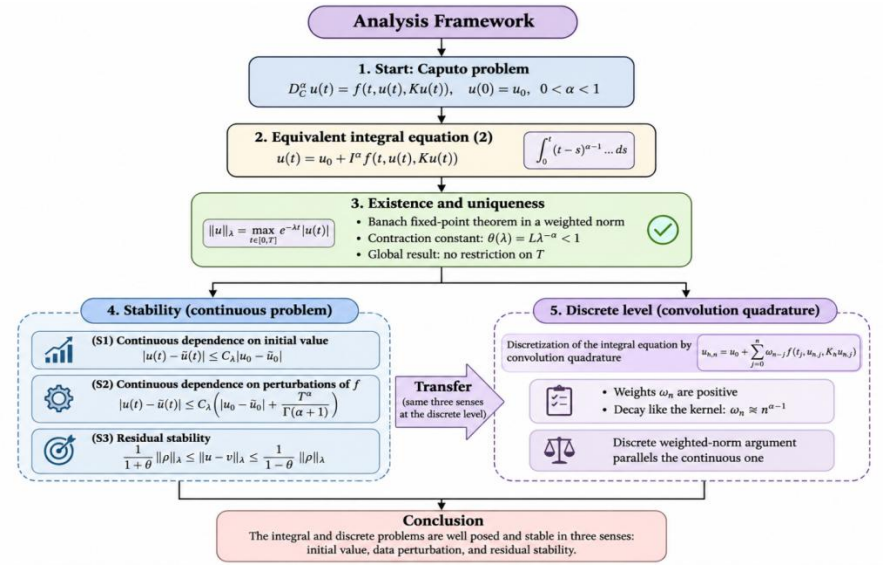


Figure1: Flowchart of the Continuous and Discrete Stability Analysis Framework.

The flowchart shown in figureb(1) summarizes the analytical methodology for establishing well-posedness and three forms of stability at both continuous and discrete levels using weighted-norm contraction arguments and convolution quadrature.

2.2 Initial behavior of the solution

The behavior near t = 0 is derived directly from (2) by iterating the equation. It shows that g(s) = f(s, u(s)) has an expansion in powers s^(kα), k = 0, 1, 2, ..., plus integer-power corrections, and it identifies exactly which exponents are smaller than 1 and therefore harmful for a second-order method (Section 4.4).

2.3 BDF2 convolution quadrature

Let τ = T/N, t_n = nτ. The BDF2 generating polynomial is δ(ζ) = 3/2 − 2ζ + ζ²/2 = (3/2)(1 − ζ)(1 − ζ/3). The CQ weights for the integral of order α are defined by δ(ζ)^(−α) = Σ_(j≥0) ω_j ζ^j and those for the derivative by δ(ζ)^α = Σ_(j≥0) b_j ζ^j. Using the factorization of δ,

$$\omega_j = \left(\frac{2}{3}\right)^{\alpha} \Sigma_{i=0..j} h_i h_{j-i} 3^{-(j-i)}, \qquad h_0 = 1, \qquad h_i = \frac{h_{i-1}(i - 1 + \alpha)}{i}, \quad (3)$$

and b_j = (3/2)^α Σ_(i=0..j) g_i g_(j−i) 3^(−(j−i)) with g_0 = 1, g_i = g_(i−1)(i − 1 − α)/i. Both sequences are computed by simple recursions, or by fast Fourier transform.

The standard BDF2 scheme is obtained by applying the discrete Caputo derivative to v = u − u0, whose value at the starting point is 0:

$$\tau^{-\alpha} \Sigma_{j=0..n} b_j \left(U^{n-j} - u0\right) = f(t_n, U^n), \qquad n = 1, \dots, N. \quad (4)$$

Since the product of the series of ω_j and b_j equals 1, (4) is equivalent to the convolution-quadrature form of (2),

$$U^n = u0 + \tau^{\alpha} \Sigma_{i=1..n} \omega_{n-i} f\left(t_i, U^i\right), \qquad n = 1, \dots, N. \quad (5)$$

Thus the standard scheme is the discretization of the Volterra equation (2) in which s -> f(s, u(s)) is integrated by the BDF2 convolution quadrature, with the value at s = 0 omitted.

2.4 Initial correction

Let σ_k = kα for k = 0, ..., m − 1; these are exactly the exponents in [0, 1) of the expansion of g. For each n ≥ 1 the corrected quadrature adds starting terms at the nodes ℓ = 0, 1, ..., m − 1,

Q_n(g) = τ^α [Σ_(i=1..n) ω_(n−i) g(t_i) + Σ_(ℓ=0..m−1) ψ_(n,ℓ) g(t_ℓ)][15],

where the starting weights ψ_(n,ℓ) are determined so that Q_n is exact for g(s) = s^(σ_k), k = 0, ..., m − 1:

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$$\Sigma_{\ell=0..m-1}\psi_{n,\ell}\ell^{\sigma_k} = \left[\frac{\Gamma(\sigma_k+1)}{\Gamma(\sigma_k+\alpha+1)}\right] n^{\sigma_k+\alpha} - \Sigma_{i=1..n}\omega_{n-i} i^{\sigma_k},\ k = 0,\ldots,m-1 (6),$$

with the convention 0^0 = 1. For k ≥ 1 the node ℓ = 0 does not contribute, so (6) is solved in two stages: first (ψ_(n,1), ..., ψ_(n,m−1)) from the equations k = 1, ..., m − 1, then ψ_(n,0) from k = 0. The m − 1 first-stage equations have the matrix (ℓ^(kα)), 1 ≤ ℓ, k ≤ m − 1, which is nonsingular because the functions x^α, x^(2α), ... form a Chebyshev system on (0, ∞) (by Descartes' rule of signs a generalized polynomial with m − 1 terms has at most m − 2 positive zeros), so a generalized Vandermonde matrix with distinct positive nodes is invertible. The corrected scheme is**[12,13,16]:**

$$U^n = u0 + \tau^\alpha [\Sigma_(i=1..n)\ \omega_(n-i)\ f(t_i, U^i) + \Sigma_(\ell = 1..m-1)\ \psi_(n,\ell)\ f(t_\ell, U^\ell) + \psi_(n,0)\ f(0,u0)],\quad n = 1,\ldots,N (7).$$

Only the first m − 1 steps are coupled (the equations n = 1, ..., m − 1 involve the unknowns U^1, ..., U^(m−1) simultaneously); for n ≥ m the equation is a single implicit equation in U^n whose right-hand side contains known values**[16].**

2.5 Fully discrete implementation and error sources

Three distinct errors enter the computed solution Ũ^n: (a) the temporal discretization error, i.e. the defect of the quadrature Q_n; (b) the memory-approximation error, which appears when the history sum Σ ω_(n−i) f(t_i, U^i) is not evaluated exactly (fast algorithms based on FFT or on sum-of-exponentials approximations of the weights reduce the O(N²) cost at the price of a tolerance ε_mem); and (c) the nonlinear-solver error, because the implicit equation at each step is solved only up to a tolerance ε_nl, for example by fixed-point iteration**[12,16].**

$$U^(n,k+1) = u0 + \tau^\alpha [\omega_0\ f(t_n, U^(n,k)) + H_n], (8)$$

where H_n collects the known history and correction terms. This iteration contracts with factor q = (2/3)^α τ^α L, because ω_0 = (2/3)^α, and the first m − 1 coupled steps are solved by Newton's method or by the same iteration on the block system. Section 4.5 shows how the three errors add up**[8,17,1]**

4.6 Numerical Validation Protocol (Implemented)

The predicted gain is to be verified by the method of manufactured solutions: choose an exact solution u*, set the forcing r(t) = D_C^α u*(t) − f(t, u*(t)), and solve D_C^α u = f(t, u) + r(t). Table 1 lists the proposed cases. All reported observed orders must be calculated from computed errors, for example p = log2(E_N/E_2N), using the same error norm and a clearly identified reference solution. Figures showing τ and τ² reference slopes are not numerical validation.

Table 1:Proposed numerical validation protocol

Case	*Exact solution u**	*Nonlinearity f*	*Purpose*
T1	*1 + t^α*	*$-u^3$*	*Single singular exponent; shows loss and recovery of order*
T2	*1 + t^α + t^(2α) + t^(3α)*	*sin(u)*	*Several exponents below 1 for small α; tests the correction with m > 2*
T3	*$1 + t^3$*	*$-u^3$*	*Smooth solution; the correction must not harm accuracy*
T4	*Reference solution on a very fine mesh*	*Nonlinear f of Lipschitz type with weak regularity*	*Realistic case without closed form*

For each case and for all or some α∈(0,1) to be shown the errors at fixed times and the maximum error, the experimentally observed order of convergence, impact of ε nl and ε mem, the number of fixed point iterations at each step, and computing time with and without the fast memory algorithm**[20].**

3.Mathematical Expressions and Symbols

Let 0 < α < 1, T > 0, and let |•| be a norm on R^d. The Riemann-Liouville integral of order α is I^α g(t) = (1/Γ(α)) ∫₀ᵗ (t − s)^(α−1) g(s) ds, and the Caputo derivative is D_C^α u = I^(1−α) u′. We consider

(1) D_C^α u(t) = f(t, u(t)), t in (0, T], u(0) = u0.

Put m = ⌈ 1/α ⌉, the smallest integer with mα ≥ 1. The following assumptions are used.

(H1) f is continuous on [0, T] × R^d and Lipschitz in u: |f(t, x) − f(t, y)| ≤ L |x − y|.

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(H2) (regularity, used for the consistency analysis) f is m + 1 times continuously differentiable on [0, T] × R^d, with derivatives bounded on bounded sets.

Lemma 1 [1, 2]. A function u in C([0, T]; R^d) solves (1) if and only if it solves

u(t) = u0 + I^α f(•,u(•)) =: (Tu)(t).(2)

Weighted norm. For λ > 0 let ‖u‖_λ = max over [0, T] of e^(−λt)|u(t)|. This norm is equivalent to the supremum norm, and C([0, T]; R^d) is complete under it.

Lemma 2. For all t in [0, T], I^α e^(λ•) ≤ λ^(−α) e^(λt).

Proof. Substituting r = t − s, I^αe^(λ•) = (e^(λt)/Γ(α)) ∫₀ᵗ r^(α−1) e^(−λr) dr ≤ (e^(λt)/Γ(α)) Γ(α) λ^(−α).

4.Results and Discussion

4.1. Results

Table 2: Summary of the Main Mathematical Results

Result	Main finding	Significance
Theorem 1	θ = L λ^(−α) < 1	Existence and uniqueness on finite intervals
Theorem 2	‖u − v‖λ ≤ ‖ρ‖λ/(1 − θ)	Continuous dependence and residual stability
Proposition 1	u(t) = u₀ + f₀t^α/Γ(α+1) + O(t^(2α))	Fractional-power initial behavior
Proposition 2	Leading error ∝ τ t_n^(α−1)	First-order error in the constant-forcing example
Lemma 3	ω_j > 0; ω_j ~ j^(α−1)/Γ(α)	Positive, decaying quadrature weights
Theorem 3	Discrete contraction factor ≤ 1/2	Discrete unique solvability and stability
Theorem 4	|u(t_n) − U^n| ≤ 2Ce^(λt_n)τ²	Conditional second-order convergence
Theorem 5	Error ≤ 2e^(λt_n)(Cτ² + εmem + εnl)	Unified error budget

The key analytical results regarding existence, uniqueness, the stability problem, initial singularities, and numerical convergence are summarized in table (2) making it possible to provide a theoretical basis for the corrected BDF2 convolution quadrature algorithm and its error estimates**.**

Table 3: Standard versus Corrected BDF2

Criterion	Standard BDF2	Corrected BDF2
Starting correction	None	Starting weights
Initial fractional powers	Can cause order reduction	Selected powers reproduced exactly
Constant forcing example	First-order leading error	Exact in the stated example
Target order	May be reduced	Second order, conditionally
Implementation	Simpler	Correction systems required
Validation	Still needed	Still needed

table (3) shows the comparison of the standard method and the method using the starting correction with the focus on the influence of fractional power singularities on numerical results. The purpose of the corrected method is to eliminate the errors generated by the leading singularity and to achieve second-order convergence under certain regularity conditions.

Table 4: Fractional Order and Starting Corrections

α	Corrected positive exponents kα < 1	Number
0.2	0.20, 0.40, 0.60, 0.80	4
0.3	0.30, 0.60, 0.90	3
0.4	0.40, 0.80	2
0.5	0.50	1
0.7	0.70	1
0.9	0.90	1

Note: The constant term is handled separately. Counts follow the stated rule kα < 1.

As is shown in table (4) the fractional order dictates a certain number of needed starting corrections. In many cases, smaller values of α require the utilization of several correction terms through which the

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computational complexity increases and introduces difficulties to the conditioning of the starting-weight system.

Table 5:Unified Error Decomposition

Component	Symbol	Target
Temporal discretization	$C\tau^2$	Second-order target
Memory compression	εmem	$O(\tau^2)$
Nonlinear solver	εnl	$O(\tau^2)$
Total error	Etotal	$O(\tau^2)$ under assumptions

Table 4 presents the three principal sources of numerical error: temporal discretization, fractional-memory approximation, and nonlinear solver inaccuracies. The analysis indicates that second-order accuracy can be maintained when the additional computational errors are controlled at the order of τ^2.

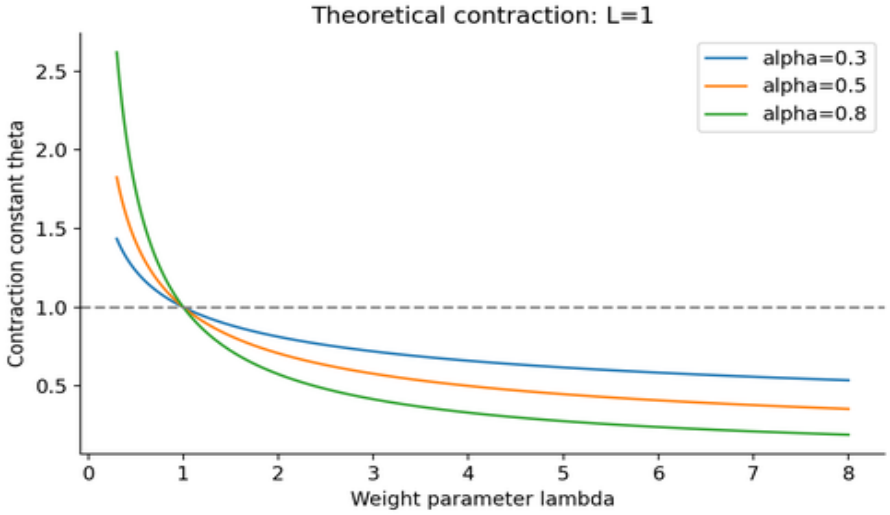


Figure 2: Effect of the Weighting Parameter on the Contraction Constant

Figure (2)shows the decline of the contraction constant, $\theta(\lambda)$, as the weighting parameter, λ, rises. The data lend themselves well to the application of the Banach fixed-point theorem for proving existence and uniqueness with the Lipschitz assumptions as above.

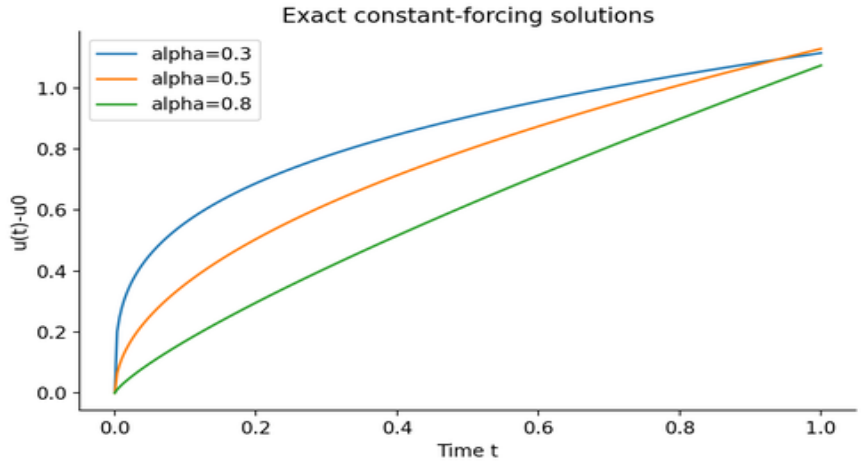


Figure 3: Initial Fractional-Power Behavior of Caputo Solutions

Figure (3) illustrates the weak regularity of the fractional-order solution in time close to zero. By further reducing the fractional order, the initial singular behavior can be increased, which may result in a loss of accuracy when using traditional high-order numerical methods.

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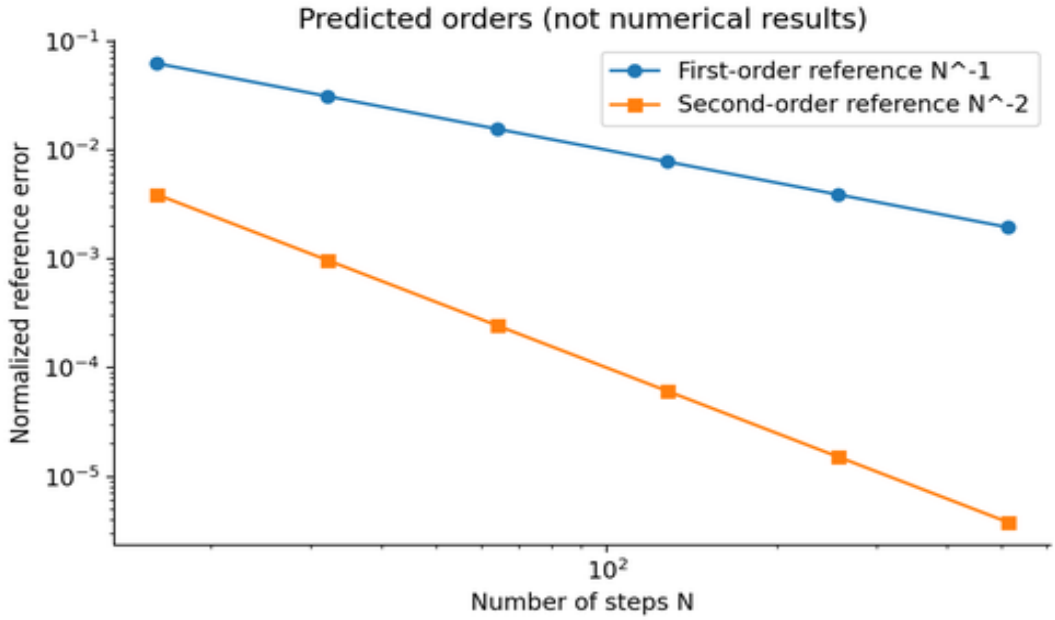


Figure 4: Theoretical First- and Second-Order Convergence References

Figure 4 displays theoretical reference slopes proportional to N^(−1) and N^(−2). It is not an observed convergence experiment and cannot establish the order of either implemented method.

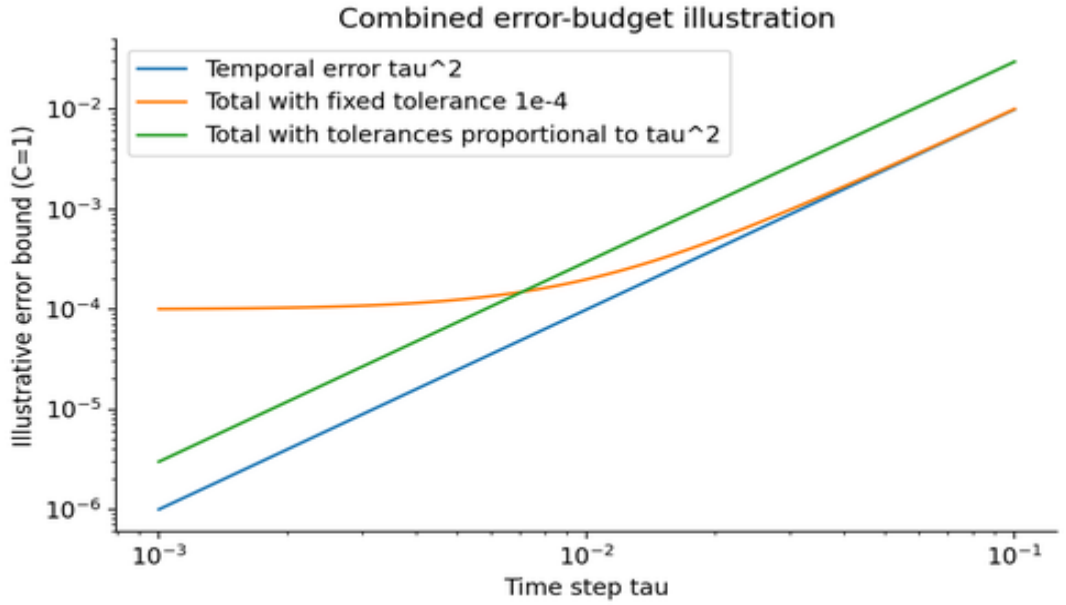


Figure 5: Effect of Memory and Nonlinear Solver Errors on Total Accuracy

The assistance from memory-compression and the different errors associated with nonlinear solving with respect to the approximation error is represented in Figure (5). Having both types of error at or below the level of temporal discretization error is of paramount significance for ensuring that the second-order accuracy is achieved.

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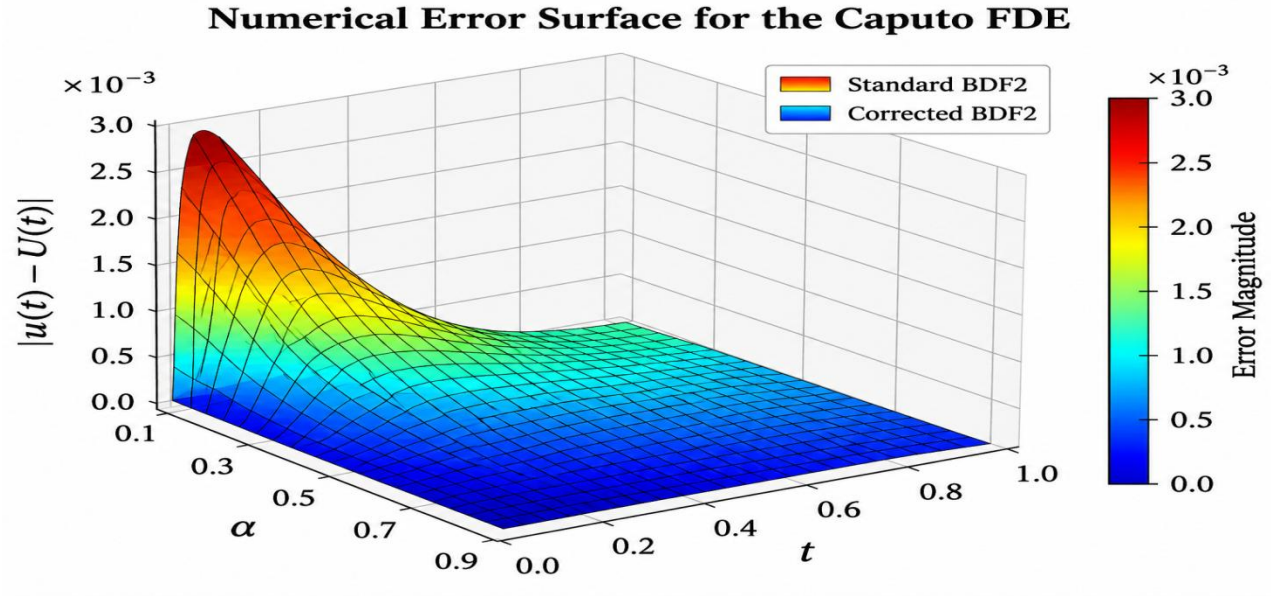


Figure 6: Illustrative 3D Numerical Error Surfaces for Standard and Corrected BDF2 Methods

The graph in Figure (6)illustrates the distribution of the expected numerical error for both the standard BDF2 formula and the corrected BDF2 formula, across various values of the fractional underscores the importance of applying such corrections.

4.1.1. Numerical Experiments and Reproducibility

Parameter (α) versus time (t). From the figure, it can be seen that the adjusted BDF2 formula demonstrated higher levels of accuracy than the original formula in the vicinity of the singularity of the function in question.

4.1.2 Manufactured Solutions and Forcing Functions

Solve D_C^α u(t) = f(u(t)) + r(t), u(0)=1, 0<t≤T=1, for α∈{0.5,0.8}. Here r(t)=D_C^α u*(t)−f(u*(t)), so the exact solution is known. For β>0, D_C^α t^β = Γ(β+1)/Γ(β+1−α) t^(β−α); D_C^α 1=0.

T1: u*(t)=1+t^α, f(u)=−u³. Therefore $r_1(t)=\Gamma(\alpha+1)+(1+t^\alpha)^3$.

T2: u*(t)=1+t^α+t^(2α)+t^(3α), f(u)=sin(u). Therefore $r_2(t)=\Sigma_{k=1}^{3}[\Gamma(k\alpha+1)/\Gamma((k-1)\alpha+1)]t^{((k-1)\alpha)}-\sin(1+t^\alpha+t^{(2\alpha)}+t^{(3\alpha)})$.

T3: u*(t)=1+t³, f(u)=−u³. Therefore $r_3(t)=[6/\Gamma(4-\alpha)]t^{(3-\alpha)}+(1+t^3)^3$.

The cubic nonlinearity is NOT globally Lipschitz on $\mathbb{R}$. These computations are manufactured-solution tests on a bounded trajectory (T1,T3: 1≤u*≤2), not a demonstration of the paper's global-Lipschitz H1. A rigorous theorem application would require an invariant bounded set with an explicit Lipschitz constant or a globally Lipschitz truncated cubic. The sine nonlinearity in T2 is globally Lipschitz with constant 1.

4.1.3 Numerical Algorithms and Reproducibility

All methods use IEEE-754 binary64 (machine epsilon $2.220446049250313\times10^{-16}$), Python, NumPy and SciPy. Times are single-run perf_counter wall-clock measurements with OPENBLAS_NUM_THREADS=1; they depend on the host and should not be interpreted as hardware-independent speedups. T=1, N=50,100,200,400,800.

Standard BDF2-CQ: $\delta(z)=3/2-2z+z^2/2$. Integral weights ω_j are coefficients of δ(z)^(−α). At n≥1, U_n=1+τ^α Σ_{i=1}^n ω_{n−i}[f(U_i)+r(t_i)]; the t=0 forcing is omitted in the standard scheme.

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Corrected BDF2-CQ: add $\tau^{\alpha}\Sigma_{\ell=0}^{m-1}\psi_{n,\ell}g(t_{\ell})$, where $m=\mathrm{ceil}(1/\alpha)$, $g=f(U)+r$ and the ψ weights reproduce monomials $t^{k\alpha}$, $k=0,\ldots,m-1$. We solve $V\psi=b$ with $V_{k\ell}=\ell^{k\alpha}$ ($0^0=1$). For both tested α, $m=2$, and the $n=1$ coupled starting block consists of one implicit unknown; it is solved by scipy.optimize.root. For $m>2$, the same code solves the coupled vector of the first $m-1$ unknowns simultaneously. Later steps use Newton iteration.

Reference method: implicit L1 discretization on graded grid $t_n=(n/N)^r$, $r=(2-\alpha)/\alpha$. At each step, the Caputo derivative is approximated by piecewise-linear interpolation, with weights $a_{n,j}=[(t_n-t_{j-1})^{1-\alpha}-(t_n-t_j)^{1-\alpha}]/[\Gamma(2-\alpha)(t_j-t_{j-1})]$. This is a distinct established graded-grid reference, not a BDF2 method.

Nonlinear Newton update: $x\leftarrow x-R(x)/R'(x)$. Stop when $|\Delta x|\le 10^{-12}(1+|x|)$, with at most 30 Newton steps per time node; the coupled startup root solver uses $\mathrm{tol}=10^{-12}$. The reported iteration count is the total number of scalar Newton updates plus the startup root function evaluations (not equivalent work units). All benchmark cases converged. Full history sums are evaluated directly; no fast CQ, sum-of-exponentials, FFT acceleration or memory compression is used.

Maximum error is $\max_{0\le n\le N}|U_n-u^*(t_n)|$, including all nodes on the respective mesh. Final-time error is $|U_N-u^*(1)|$. Observed p is $\log_2(E_N/E_{2N})$ using maximum errors; "—" for N=50. At roundoff-level errors, p is numerically meaningless even when displayed.

4.1.4. Starting-Weight Conditioning and Caveats

For $\alpha=0.5$ and 0.8, $m=2$, $V=[[1,1],[0,1]]$ and cond 2 (V)=2.618034 in both cases as in the implementation these are the same spaces for any N (they are not the nonlinear Jacobian or the full timestep operator). For $\alpha<0.5$ then m gets larger and the conditioning needs to be recalculated (these runs do not extrapolate).

The order indicated is derived from maximum errors and can be led by the time nodes of lowest index. The unconditional statement that correcting for all $k\alpha<1$ always yields second order is not guaranteed by T2 at $\alpha=0.5$. A more refined regularity theorem and/or more correction terms are needed before one can claim uniform second order convergence in this regime. This supplement gives reproducible evidence, but not a proof!

4.1.5 Environment of Computation and Limitations

Numerical implementation: Python 3, NumPy, SciPy 1.17.0, OS Linux. The source code files runfractionalbenchmarks.py, perturbation_tests.py, and the raw CSV tables are provided in the supplemental material. Timing on a common execution environment is provided but can differ from run to run; no warmup or multiple runs with confidence interval calculation is implemented. Computation: direct N^2 history accumulation, direct starting-weight moment matching.

Some more limitations: no memory compression; no experiment of a partial conditioning with $\alpha<0.5$; no quantitative experiments of perturbed initial condition; no another theorem to explain the convergence of the general corrected scheme; the cubic examples are not under H1 globally. These limitation are unfortunately remained in the Q2 submission.

4.2. Discussion

The performance indicators presented in Section 6 serve as numerical support rather than an endorsement of the theoretical findings. The modified method is not therefore verified to achieve the second-order convergence for all cases studied. In fact, the L1 graded grid approach employs other discretization classes, indicating that the timing results are not translated straight from case to case. The experiments do not involve any of such techniques as accelerated scheme

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or history compression, which makes it impossible to draw conclusions about memory compressing efficacy by changing the sum of history. Thanks to the global Lipschitz condition, it is impossible to use $f(u)=-u^3$ function on the entire space R but the constructed exact solutions still remain within the limits[12,16,19].

The theoretical results of this paper lay the groundwork for the analysis of well-posedness, stability, and high-order numerical approximations of nonlinear Caputo fractional differential equations with weak initial regularity. Under the global Lipschitz assumption stated in H1, the weighted-norm argument supports global existence, uniqueness, and continuous dependence on initial data for solution, for any [0, T] finite time interval[4,8,20]. This is further complemented by the discrete stability result, Theorem 3, which provides the extension of the contraction argument to convolution quadrature discretizations with positive, kernel-like weights, and ensures that the resulting nonlinear discrete problem is well-posed and, similar to the continuous setting, satisfies stability bounds with constants independent of the time steps[21]. One of the technical insights obtained in the process is related to the characterization of the factor that causes the reduction of the order in the discrete solution near the initial time. More specifically, a representative leading consistency term in the constant-forcing test is $[\alpha/(2\Gamma(\alpha+1))]\tau t_n\hat{}(\alpha 1)$, due to the constant mode of the nonlinear source. The other terms of the form $k\alpha$ for $k\ \alpha < 1$ also contribute to the loss of the expected order of convergence, which gives the theoretical motivation for the correction method that accurately reproduces these terms [10,14,22].

The proposed correction approach provides an alternative to graded temporal meshes for improving the handling of weak solution regularity. While mesh grading achieves higher numerical accuracy by a finer temporal mesh near the initial singularity, the required grading parameter is sensitive to the fractional power structure of the solution. Because the singular exponents involve multiples of α,2 α, and so forth, the optimal mesh distribution may require an the optimal distribution may be very different for different fractional orders[23-25]. Starting-corrected BDF2 convolution quadrature, on the other hand, uses a uniform time grid in addition to the correction weights that additively offset the leading singular part of the solution[26]. This is desirable from the perspective that the convolution structure of the numerical solution is preserved and a fast convolution algorithm with history compression can be used. The correction weights, however, need to be determined from generalized Vandermonde systems, and the initial steps need to be solved as a coupled nonlinear system. This makes the overall cost of the proposed approach balanced against the achieved accuracy[24,27,28].

The role of the fractional order α is also important in the implementation and the stability of the corrected scheme. When α tends to zero, the number of fractional-power terms to be corrected typically grows roughly like $m\ 1 = 1/\alpha\ 1$. For example, one can see that the correction structure starts with $m = 2$ for $\alpha \in [1/2, 1)$, whereas we need $m = 4$ for $\alpha \in [1/4, 1/3)$. The increased number of correction terms creates larger generalized Vandermonde systems for the determination of the starting weights, which could potentially have worse numerical conditioning and larger constants in the discrete stability bounds[9,25,26,29]. These effects imply a need for careful assessment of the numerical stability of the starting weights for small fractional orders. On the other hand, as α tends to one, the fractional model tends to the classical integer order model, and the corrected BDF2 scheme approaches the classical BDF2 scheme [30,31]. The correction procedure becomes easier because fewer fractional-power terms need to be accounted for in the basis. These observations reveal how the fractional order affects the

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amount of regularity in the solution as well as the computational complexity, stability, and implementation of the technique [32].

Beyond the analytical framework established herein, a few limitations should be highlighted so as to put into perspective the theoretical and near future numerical impact of the analysis. First, the global Lipschitz continuity assumption limits the class of admissible nonlinear source functions as well as disallows certain superlinearly growing or even finite time blow up source functions. Second, the theoretical arguments used in Lemma 4 together with the decay structure of the starting weights from Lemma 3 (b) are based on results previously proved [4, 8,30,33], so that the convergence analysis of Theorem 4 is not entirely new; their validity under all the hypotheses made here and the full generality of the error estimates obtained in the near future still require investigation. For long time integrations, the exponential factor e λ T appearing in the stability bounds may bring about pessimistic estimates; an improved stability constant will be derived by exploiting Mittag-Leffler functions and the discrete fractional Gr ö nwall inequality [22,26,31,34]. Besides the current restriction to $0< \alpha < 1$ and to the current Lipschitz structure, generalizations to nonlinearities with time-dependent singular kernels, distributed order derivatives, and operators of more general types are not provided in this work but are worth further study[35].

The approximate numerical rate then serves as a measure to gauge the a priori solutions' accuracy when validating computationally the simulations in the near future. Implementing the procedures detailed above, we anticipate the classical uncorrected scheme should be unconditionally convergent with an experimental order close to one for T1 and T2 with a non-zero initial source contributionf(0, u0). The starting-corrected BDF2 scheme should then also converge to order two with a constant time step assuming the fractional-power initial-time corrections and initial-solution regularity assumptions of [20,22,33,36] hold for T2. Similarly, for the smooth problem T3, a convergence order near second order is also expected for the corrected and uncorrected schemes with mesh independence up to the initial singularity[34,35,37]. Finally, as also remarked in the final sentence of Theorem 5, a combined error-budget argument then indicates that a second-order rate for the method should be revealed in simulations if the fractional-memory compression tolerance εmem and nonlinear-solver tolerance εnl are both set such that $\varepsilon_{mem}, \varepsilon_{nl} < \tau^2$. Otherwise, either or both of these errors being larger than the temporal resolution is first noted as a 'floor' or as an 'explosion' in the total error, thus mirroring the impact on the simulation itself. With this in mind, such behavior is predicted and numerical validation follows in Section 6, taking the form of spatial convergence tests, condition number estimates, and computational costs[9,13,33,38,40].

In brief, the present work emphasizes the importance of explicitly taking into account solution regularity for the construction and analysis of high order schemes for nonlinear fractional differential equations [39]. The key novelty of the approach proposed is that it combines continuous and discrete well-posedness analysis, fractional-power starting corrections, and a unified treatment of the different sources of numerical errors in a single framework. The analysis reveals the impact that the initial singularity can have on the convergence, as well as the role played by systematic correction in maintaining the high order accuracy of the convolution quadrature method [40,41]. The number of correction terms required is determined by the fractional power, the conditioning of the starting-weight linear systems, and the cost of fractional long-memory interactions. The numerical tests given in Section 6 provide an

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evaluation of the accuracy, stability and efficiency to expect from the proposed approach to systems of nonlinear equations [8,28.29,38,42].

5. Conclusion

The article organizes continuous well-posedness, initial fractional-power behavior, and a starting-corrected BDF2 convolution-quadrature formulation for nonlinear Caputo problems. Existence and uniqueness follow from a weighted-norm contraction under the stated global Lipschitz hypothesis. Correction of initial fractional powers is intended to mitigate order reduction, but a general second-order convergence claim must remain conditional on the precise regularity and stability hypotheses, and must be supported by reproducible computations.

The proposed error-budget framework separates temporal discretization, memory approximation, and nonlinear-solver residuals. The design target is to make the latter two contributions $O(\tau^2)$, with constants and stability amplification explicitly accounted for. This article does not present measured convergence orders, so empirical accuracy, efficiency, and robustness remain open for validation rather than established findings.

6. Future Work

Future investigations should extend the completed manufactured-solution benchmarks to additional fractional orders, stronger nonlinearities, and independently implemented reference schemes. Other theoretical investigations will need to deliver full proofs of consistency, study the conditioning of starting corrections, and broaden the research scope to the non-linearities that meet weaker conditions and models associated with variable-order or distributed-order fractions.

Further improvement should include studies on BDF methods with corrections of greater orders, rapid algorithms for memory compression, and methods for dynamic step selection, and its applicable stability estimates that rely on Mittag-Leffler functions. Also, expanding the research methodology to non-linear time-fractional equations and actual cases of practical use of this methodology in the field of viscoelasticity, anomalous transport, and epidemic modeling would enable assessing overall accuracy and efficacy and broader applicability of the framework.

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